THE DYNAMIC EQUATIONS OF MOTION FOR A TWO-COMPONENT SYSTEM

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The motion of solid particles in a fluid flow is represented as a random process with independent increments. The resulting kinetic equation for the particle distribution has the form previously proposed [1]. The solution to this equation provides a system of equations for the hydrodynamics of the assembly of solid particles. These equations differ from ones previously proposed [2, 3] in having additional terms related to relative motion of the components, whose presence is due to anisotropy in the distribution of the normal stresses in the pseudogas.

§1. **Derivation of the kinetic equation.** Consider the motion of a large set of particles in a flow of fluid. The speed of a particular particle is influenced by forces of three types: external mass forces, interaction between the particle and the flow, and collisions.

Each particle perturbs the carrier and thus affects the interactions of the other particles, so the motion of one particle will, in general, be dependent on the motion of all.

The numerous random factors will result in a relatively smooth and slow variation in the speed of the flow. Direct collisions between particles will also affect the velocities, but such interactions are essentially short-range ones, whereas the others are not.

If the fluid is of low viscosity, particle collisions may be considered as statistically independent.

Let $\{u, x\}$ be the radius vector of the point representing the state of a system of N particles in phase space:

$$\{\mathbf{u}, \mathbf{x}\} = \{\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(N)}; \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(N)}\}.$$

Here $u^{(i)}$ is the velocity vector of particle i, whose center of mass has radius vector $x^{(i)}$ relative to a fixed Cartesian coordinate system.

The following assumptions are made:

1. The velocity vector u(t) of the point in phase space may be considered as a random variable with independent increments.
2. Particle collisions may be represented as collisions of elastic spheres.
3. We can neglect simultaneous collisions of three or more particles.

The last assumption is largely a consequence of the previous one and can be taken as applying to any short-range interaction and any number of particles, even when close packing is approached [4].

A random process with independent increments can be represented as the sum of two random processes: a continuous diffusion and a random process composed of steps in the initial process. To determine the conditional probability density $\Psi(t, x, u)$ for transition from one state to another we have [5]

$$\frac{\partial \Psi}{\partial t} = -\sum_{i=1}^{N}\sum_{\alpha=1}^{3} u_\alpha^{(i)} \frac{\partial \Psi}{\partial x_\alpha^{(i)}} + \sum_{i=1}^{N}\sum_{\alpha=1}^{3} \frac{\partial}{\partial u_\alpha^{(i)}}\left[a_\alpha^{(i)}\Psi + \sum_{\beta=1}^{3} B_{\alpha\beta}^{(i)} \frac{\partial \Psi}{\partial u_\beta^{(i)}}\right] +$$

$$+ \sum_{1\leqslant i<j\leqslant N} \int_{(1)} [\Psi(t, \mathbf{x}, A_{ij}(\mathbf{l})\mathbf{u}) - \Psi(t, \mathbf{x}, \mathbf{u})]\, \varphi_{ij} d\mathbf{l},$$

$$A_{ij}(\mathbf{l})\mathbf{u} = \{\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(i-1)}, \mathbf{u}^{(i)} + \mathbf{l}\sum_{\alpha=1}^{3} l_\alpha (u_\alpha^{(j)} - u_\alpha^{(i)}), \mathbf{u}^{(i+1)}, \ldots$$

$$\ldots, \mathbf{u}^{(j-1)}, \mathbf{u}^{(j)} - \mathbf{l}\sum_{\alpha=1}^{3} l_\alpha (u_\alpha^{(j)} - u_\alpha^{(i)}), \mathbf{u}^{(j+1)}, \ldots, \mathbf{u}^{(N)}\}, \qquad (1.1)$$

in which $A_{ij}(\mathbf{l})$ is the operator for transition between states when particle i collides with particle j, $\mathbf{l}$ is the unit vector specifying the direction of the line between the centers of those particles, and $\varphi_{ij}(\mathbf{x}, \mathbf{u})$ is the probability density for collision of these particles (probability of $\varphi_{ij} d\mathbf{l} dt$ of collision in unit time).

The quantities $a_\alpha^{(i)}$ and $B_{\alpha\beta}^{(i)}$ characterize the continuous component of the process with independent increments.

The following distribution functions are introduced:

$$f(\mathbf{u}^{(1)}, \mathbf{x}^{(1)}, t) = \frac{1}{V^{N-1}} \int \Psi(t, \mathbf{x}, \mathbf{u})\, d\mathbf{u}^{(2)} \ldots d\mathbf{u}^{(N)} d\mathbf{x}^{(2)} \ldots d\mathbf{x}^{(N)},$$

$$g(t; \mathbf{u}^{(1)}, \mathbf{x}^{(1)}; \mathbf{u}^{(2)}, \mathbf{x}^{(2)}) = \frac{1}{V^{N-2}} \int \Psi(t, \mathbf{x}, \mathbf{u})\, d\mathbf{u}^{(2)} \ldots d\mathbf{u}^{(N)} d\mathbf{x}^{(3)} \ldots d\mathbf{x}^{(N)}. \qquad (1.2)$$

The integrations in (1.2) are carried out over all permissible values of the variables. The procedure of (1.2) is applied to (1.1) to give

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{3} u_\alpha^{(1)} \frac{\partial f}{\partial x_\alpha^{(1)}} = \sum_{\alpha=1}^{3} \frac{\partial}{\partial u_\alpha^{(1)}}\left[a_\alpha f + \sum_{\beta=1}^{3} B_{\alpha\beta} \frac{\partial f}{\partial u_\beta^{(1)}}\right] +$$

$$+ \iint g(t; \mathbf{x}^{(1)}, \mathbf{u}_*^{(1)}; \mathbf{x}^{(1)} + \mathbf{l}\sigma; \mathbf{u}_*^{(2)})\, K_* d\mathbf{l}\, d\mathbf{u}_*^{(2)} -$$

$$- \iint g(t; \mathbf{x}^{(1)}, \mathbf{u}^{(1)}; \mathbf{x}^{(1)} + \mathbf{l}\sigma; \mathbf{u}^{(2)})\, K d\mathbf{l}\, d\mathbf{u}^{(2)}, \qquad (1.3)$$

in which K is the collision cross section, while the quantities $\mathbf{u}_*^{(1)}$, $\mathbf{u}_*^{(2)}$, $\mathbf{u}^1$, and $\mathbf{u}^2$ are related via the operator $A_{12}(\mathbf{l})$:

$$\mathbf{u}_*^{(1)} = \mathbf{u}^{(1)} + \mathbf{l}\sum_{\alpha=1}^{3} l_\alpha (u_\alpha^{(2)} - u_\alpha^{(1)}),$$

$$\mathbf{u}_*^{(2)} = \mathbf{u}^{(2)} - \mathbf{l} \sum_{\alpha=1}^{3} l_\alpha (u_\alpha^{(2)} - u_\alpha^{(1)}). \quad (1.4)$$

We now apply the hypothesis of molecular chaos:

$$g(t; \mathbf{x}^{(1)}, \mathbf{u}^{(1)}; \mathbf{x}^{(2)}, \mathbf{u}^{(2)}) = f(\mathbf{x}^{(1)}, \mathbf{u}^{(1)}, t) f(\mathbf{x}^{(2)}, \mathbf{u}^{(2)}, t). \quad (1.5)$$

This applies for dense suspensions within this collision model [4].

Then (1.3) and (1.5) give an equation for f:

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{3} u_\alpha \frac{\partial f}{\partial x_\alpha} = \sum_{\alpha=1}^{3} \frac{\partial}{\partial u_\alpha}\left[a_\alpha f + \sum_{\beta=1}^{3} B_{\alpha\beta} \frac{\partial f}{\partial u_\beta}\right] +$$

$$+ \sigma^2 \iint [\chi' f' f_1' - \chi f f_1] k d\mathbf{l} d\mathbf{u}_1, \quad (1.6)$$

where the symbols in the second term on the right-hand side are standard ones in the kinetic theory of gases.

Equation (1.6) coincides with the equation previously proposed [1].

§2. Characterization of the continuous component. The equation of motion of a particle may be put as

$$\frac{d\mathbf{u}}{dt} = \frac{1}{m}(\mathbf{G} + \mathbf{F} + \mathbf{K}), \quad (2.1)$$

in which G is the external mass force (e.g., gravity), F is the force from the interaction of the particles with the fluid, and K is the force between colliding particles, in which K(t) may be represented as a sum of δ-functions.

The characteristic time of free motion between two collisions is much less than the characteristic time for F(t), so the characteristics of the diffusion operator in (1.6) may be determined from a simplified form of (2.1):

$$d\mathbf{u}/dt = m^{-1}(\mathbf{G} + \mathbf{F}). \quad (2.2)$$

In what follows we consider only the case in which the particles are much more dense than the fluid. Then for the force exerted on a particle by the fluid we have

$$\mathbf{F}/m = \Phi(\varepsilon, |\mathbf{s} - \mathbf{u}|)(\mathbf{s} - \mathbf{u}), \quad (2.3)$$

in which s is the velocity of the fluid, u is the velocity of the particle, and ε is the mean relative volume of the fluid in a sufficiently large space around a particle, which is related to the number of particles in that volume by $\varepsilon = 1 - nv_0$, in which v_0 is the volume of a particle.

Estimates [6] show that only fairly large inhomogeneities play a major part in energy transfer from gas to particles.

A semiempirical analysis [7] indicates that we must allow for the dependence of F on ε; this agrees well with experiment.

We represent s and **u** as

$$\mathbf{s} = \mathbf{q} + \boldsymbol{\omega}, \quad \mathbf{u} = \boldsymbol{\omega} + \mathbf{v}, \quad \mathbf{w} = \int \mathbf{u} f d\mathbf{u}, \quad \int \mathbf{v} f d\mathbf{u} = 0,$$

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \boldsymbol{\omega}(\xi) d\xi = 0, \quad (2.4)$$

in which q is the mean velocity of the fluid near a particle when the mean porosity of the suspension in that region is ε.

We further assume that $|\omega|$ and $|v|$ are small relative to $|q - w|$, and put F as

$$\frac{\mathbf{F}}{m} = \Phi(\varepsilon, |\mathbf{q} - \mathbf{w}|)(\mathbf{q} - \mathbf{w}) +$$

$$+ \Phi(\varepsilon, |\mathbf{q} - \mathbf{w}|)(\boldsymbol{\omega} - \mathbf{v}) +$$

$$+ \frac{\partial \Phi(\varepsilon, |\mathbf{q} - \mathbf{w}|)}{\partial \varepsilon} \Delta\varepsilon (\mathbf{q} - \mathbf{w}), \quad (2.5)$$

neglecting terms of order $|\omega - v|^2$, $(\Delta\varepsilon)^2$, etc., where $\Delta\varepsilon$ is the fluctuation in the porosity around a particle.

Expression (2.4) can be put as

$$m^{-1} F_\alpha = -a_\alpha^* + A_\alpha \quad (\alpha = 1, 2, 3),$$

$$a_\alpha^* = -\Phi(\varepsilon, |\mathbf{q} - \mathbf{w}|)(q_\alpha - w_\alpha) + \Phi(\varepsilon, |\mathbf{q} - \mathbf{w}|) v_\alpha,$$

$$A_\alpha = \Phi(\varepsilon, |\mathbf{q} - \mathbf{w}|)\omega_\alpha + \frac{\partial \Phi(\varepsilon, |\mathbf{q} - \mathbf{w}|)}{\partial \varepsilon} \Delta\varepsilon (q_\alpha - w_\alpha). \quad (2.6)$$

The time-average of A_α is zero, so we have

$$\frac{d\mathbf{v}}{dt} = \frac{\mathbf{G}}{m} - \frac{d\mathbf{w}}{dt} - \mathbf{a}^* + \mathbf{A}. \quad (2.7)$$

For the steady state

$$\mathbf{G}/m + \Phi(\varepsilon, |\mathbf{q} - \mathbf{w}|)(\mathbf{q} - \mathbf{w}) = d\mathbf{w}/dt \quad (2.8)$$

and we have the following stochastic equation for **v**:

$$\frac{d\mathbf{v}}{dt} = \Phi(\varepsilon, |\mathbf{q} - \mathbf{w}|)(\boldsymbol{\omega} - \mathbf{v}) +$$

$$+ \frac{\partial \Phi(\varepsilon, |\mathbf{q} - \mathbf{w}|)}{\partial \varepsilon} \Delta\varepsilon (\mathbf{q} - \mathbf{w}). \quad (2.9)$$

The quantities ω and $\Delta\varepsilon$ are not independent. Moreover, $\Delta\varepsilon$ is directly related to fluctuations in the number of particles in a given volume, so the mode of fluctuation of $\Delta\varepsilon$ is determined by the statistical properties of the particle system; $\Delta\varepsilon$ is a functional of f, and so a^* and $B_{\alpha\beta}$ are functionals of f, and (1.6) resembles the kinetic equations of self-consistent fields [8]. Relation (2.9) shows that the statistical characteristics of v(t) may have pronounced anisotropy in the general case. For the isotropic state, G = q = w = 0 and

$$d\mathbf{v}/dt = -\Phi(\varepsilon, 0)\mathbf{v} + \Phi(\varepsilon, 0)\boldsymbol{\omega}, \quad \mathbf{v} = \mathbf{u}. \quad (2.10)$$

Then

$$a_\alpha = \Phi(\varepsilon, 0) u_\alpha, \qquad B_{\alpha\beta} = B_0(\varepsilon)\delta_{\alpha\beta}. \quad (2.11)$$

If the components are in relative motion, the condition for conservation of the mean flow rate of the fluid gives

$$\mathbf{v} = \mathbf{v}_1 + V \frac{\mathbf{q} - \mathbf{w}}{|\mathbf{q} - \mathbf{w}|}, \qquad \boldsymbol{\omega} = \boldsymbol{\omega}_1 - \frac{\Delta\varepsilon}{\varepsilon}(\mathbf{q} - \mathbf{w}), \quad (2.12)$$

in which ω_1 is uniformly distributed with respect to direction and $|v_1| \sim |\omega_1| \ll |V|$.

The stochastic equation for v_1 and V is

$$\frac{dV}{dt} = -\Phi(\varepsilon, |\mathbf{q} - \mathbf{w}|) V -$$

$$-\Phi\left\{\frac{1}{\varepsilon}-\frac{\partial\ln\Phi}{\partial\varepsilon}\right\}|\mathbf{q}-\mathbf{w}|\Delta\varepsilon,$$

$$\frac{d\mathbf{v}_1}{dt}=\Phi(\varepsilon,|\mathbf{q}-\mathbf{w}|)(\boldsymbol{\omega}_1-\mathbf{v}_1). \quad (2.13)$$

The result for a_α and $B_{\alpha\beta}$ is

$$a_\alpha^*=-\frac{G_\alpha}{m}-\Phi(\varepsilon,|\mathbf{q}-\mathbf{w}|)(q_\alpha-w_\alpha)+$$

$$+\Phi(\varepsilon,|\mathbf{q}-\mathbf{w}|)(u_\alpha-w_\alpha),$$

$$B_{\alpha\beta}=B_0(\varepsilon)\delta_{\alpha\beta}+B(q_\alpha-w_\alpha)(q_\beta-w_\beta). \quad (2.14)$$

The dependence of B on the parameters of the system may be deduced via the theory of stationary random processes [9]:

$$B=\Phi^2(\varepsilon,|\mathbf{q}-\mathbf{w}|)\left\{\frac{1}{\varepsilon}-\frac{\partial\ln\Phi}{\partial\varepsilon}\right\}^2\langle(\Delta\varepsilon)^2\rangle T, \quad (2.15)$$

in which T is the characteristic time of the fluctuations in $\Delta\varepsilon$, which produce changes in particle motion until the viscous forces on each particle produce a steady-state relation between $|\mathbf{q}-\mathbf{w}|$ and $\varepsilon+\Delta\varepsilon$. Then

$$T=\frac{D}{\Phi(\varepsilon,|\mathbf{q}-\mathbf{w}|)}, \quad (2.16)$$

in which D is some constant.

Substitution of (2.16) into (2.15) gives

$$B=D\Phi(\varepsilon,|\mathbf{q}-\mathbf{w}|)\left\{\frac{1}{\varepsilon}-\frac{\partial\ln\Phi}{\partial\varepsilon}\right\}^2\langle(\Delta\varepsilon)^2\rangle. \quad (2.17)$$

We can always assume that $B_0 \ll B$.

Equation (2.13) provides some general conclusions on the particle motion. If $\Delta\varepsilon < 0$ (aggregation), acceleration occurs in the direction of the relative velocity of the components. In a fluidized bed, this corresponds to rising motion of a closely linked group of particles, which is in generally good agreement with experiment [10].

Complete determination of B requires the explicit form of the dependence of $\Delta\varepsilon$ on f, and this is dependent on the properties of the solution to (1.7).

§3. Analog of the H theorem and stationary states. Consider the case of a spatially homogeneous state of the system

$$(\partial f/\partial x_\alpha)=0,\quad \mathbf{G}=\mathbf{q}=\mathbf{w}=0. \quad (3.1)$$

The kinetic equation for $f(\mathrm{u},t)$ is

$$\frac{\partial f}{\partial t}=C(ff_1)+\sum_{i=1}^{3}\frac{\partial}{\partial u_i}\left[\Phi(\varepsilon,0)u_if+B_0(\varepsilon)\frac{\partial f}{\partial u_i}\right], \quad (3.2)$$

in which $C(ff_1)$ is the collision operator; ε = const, in view of the spatial homogeneity of the state.

Theorem. The total derivative with respect to time of

$$H(t)=\int f(\mathbf{u},t)\ln\varphi(\mathbf{u},t)\,d\mathbf{u},$$

$$\varphi(\mathbf{u},t)=f(\mathbf{u},t)\exp(-\Phi u^2/2B_0) \quad (3.3)$$

is not positive, i.e., $(dH/dt)\le 0$.

Proof. We differentiate H with respect to time and integrate by parts with use of (3.2) to get

$$\frac{dH}{dt}=\int C(ff_1)(\ln\varphi+1)\,d\mathbf{u}-$$

$$-B_0\int\exp\left(-\frac{\Phi u^2}{2B_0}\right)\sum_{i=1}^{3}\frac{1}{\varphi}\left(\frac{\partial\varphi}{\partial u_i}\right)^2d\mathbf{u}. \quad (3.4)$$

The first term in (3.4) has a form familiar from the kinetic theory of gases [11], with

$$\int C(ff_1)(\ln\varphi+1)\,d\mathbf{u}\leqslant 0 \quad (3.5)$$

and with equality for an arbitrary function of the form $A\exp(-\gamma u^2)$, in which A and γ are constants. We put $g=\varphi^{1/2}$ to transform the second term in (3.4) to

$$-B_0\int\exp\left(-\frac{\Phi u^2}{2B_0}\right)\sum_{i=1}^{3}\frac{1}{\varphi}\left(\frac{\partial\varphi}{\partial u_i}\right)^2d\mathbf{u}=$$

$$=-4B_0\int\exp\left(-\frac{\Phi u^2}{2B_0}\right)|\nabla\rho|^2d\mathbf{u}\leqslant 0, \quad (3.6)$$

with equality if φ = const. The normalization condition shows that (dH/dt) = 0 for the function

$$f^{(0)}=n\left(\frac{\Phi}{2\pi B_0}\right)^{3/2}\exp\left(-\frac{\Phi u^2}{2B_0}\right). \quad (3.7)$$

The $f^{(0)}$ of (3.7) has the same form as the Maxwellian distribution in the kinetic theory of gases, the difference being that the factor to u^2 in the exponential is no longer arbitrary. This feature is entirely natural from the viewpoint of the mode of action of each of the operators on the right in (3.2).

The collision operator tends to bring the kinetic energies of colliding particles to the mean value, since the difference between the kinetic energies is nearly always reduced by collision [12], although the initial energy of the system is retained.

The diffusion operator acts by matching the energy input to the level of dissipation; the latter is dependent on the kinetic energy, so the system takes up a certain mean kinetic energy.

There are two stages in the approach to the equilibrium state: rapid equalization of the kinetic energies and then slow evolution of the mean toward some definite value. In fact, (3.2) has the following solution:

$$f=n\left(\frac{m}{2\pi\theta}\right)^{3/2}\exp\left(-\frac{mu^2}{2\theta}\right),$$

$$\theta=\frac{m}{2}\left[\frac{m}{2\theta_0}e^{-2\Phi t}+\frac{B_0}{2\Phi}(1-e^{-2\Phi t})\right]^{-1}, \quad (3.8)$$

in which θ is the effective temperature of the pseudogas, θ_0 is the value at t = 0 and n is the number of particles in unit volume.

§4. Transport equations for the pseudogas and solution of the kinetic equation. Equation (1.7) allows us to derive the transport equations in the standard way. We introduce the following means:

$$n=\int f d\mathbf{u},\quad \rho=mn,\quad w_\alpha=\frac{1}{n}\int u_\alpha f d\mathbf{u},$$

$$\theta=\frac{1}{3n}\int m(\mathbf{u}-\mathbf{w})^2 f d\mathbf{u}. \quad (4.1)$$

The result is

$$\frac{\partial \rho}{\partial t}+\sum_{\alpha=1}^{3}\frac{\partial \rho w_\alpha}{\partial x_\alpha}=0,$$

$$\rho\frac{\partial w_i}{\partial t}+\sum_{\alpha=1}^{3}\rho w_\alpha\frac{\partial w_i}{\partial x_\alpha}=$$

$$=\sum_{\alpha=1}^{3}\frac{\partial P_{i\alpha}}{\partial x_\alpha}+\rho\left[\frac{G_i}{m}+\Phi\,(q_i-w_i)\right],$$

$$\frac{\partial \theta}{\partial t}+\sum_{\alpha=1}^{3}w_\alpha\frac{\partial \theta}{\partial x_\alpha}=-\frac{2}{3n}\left[\sum_{\alpha=1}^{3}\frac{\partial Q_\alpha}{\partial x_\alpha}+\sum_{\alpha,\beta=1}^{3}P_{\alpha\beta}\frac{\partial w_\alpha}{\partial x_\beta}\right]+$$

$$+\frac{2m}{3}\,[3B_0+B]-2\Phi\theta\,. \qquad (4.2)$$

System (4.2) differs from the usual system of transport equations only in that the energy equation contains source-type terms; Q_α and $P_{\alpha\beta}$ are defined as usual in the kinetic theory of dense gases [11].

We replace u by the new independent variable c = = u − w and write (3.9) in the form

$$\frac{Df}{Dt}+\sum_{\beta=1}^{3}\left[C_\beta\frac{\partial f}{\partial x_\beta}+\right.$$

$$\left.+\left(R_\beta-\frac{Dw_\beta}{Dt}\right)\frac{\partial f}{\partial c_\beta}\right]-\sum_{\alpha,\beta=1}^{3}C_\beta\frac{\partial w_\alpha}{\partial x_\beta}\frac{\partial f}{\partial c_\alpha}-$$

$$-\sum_{i=1}^{3}\frac{\partial}{\partial c_i}\left[\Phi c_i f+\sum_{j=1}^{3}B_{ij}\frac{\partial f}{\partial c_j}\right]=C\,(ff_1),$$

$$R_\beta=\frac{G_\beta}{m}+\Phi\,(\varepsilon,|\mathbf{q}-\mathbf{w}|)\,(q_\beta-w_\beta)\,, \qquad (4.3)$$

where the operator is

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+\sum_{\alpha=1}^{3}w_\alpha\frac{\partial}{\partial x_\alpha}\,. \qquad (4.4)$$

In accordance with the Chapman-Enskog method, we seek a solution to (4.3) in the form

$$f=\sum_{r=0}^{\infty}f^{(r)}. \qquad (4.5)$$

We use as zeroth approximation the Maxwellian distribution

$$f_{(0)}=n\left(\frac{m}{2\pi\theta}\right)^{3/2}\exp\left(-\frac{mc^2}{2\theta}\right), \qquad (4.6)$$

in which $n(\mathbf{x},t)$, $w_\alpha(\mathbf{x},t)$ and $\theta(\mathbf{x},t)$ coincide with the true values of these quantities for the pseudogas.

We apply a form of Enskog's method [11] for the kinetic equation for a dense gas to show that, if we use the tensor

$$E_{ij}=\frac{\partial w_i}{\partial x_j}-\frac{mB_{ij}}{\theta\,(1+{}^4/_{15}\pi\sigma^3 n\chi)}, \qquad (4.7)$$

the equation for $f^{(1)}$ will take the form

$$\chi\,[C\,(f^{(0)}f_1^1)+C\,(f_1^{(0)}f^{(1)})]=$$

$$=f^{(0)}\left\{\left(1+\frac{2}{5}\pi\sigma^3 n\chi\right)\left(\frac{mc^2}{2\theta}-\frac{5}{2}\right)\sum_{i=1}^{3}c_i\frac{\partial\ln\theta}{\partial x_i}+\right.$$

$$\left.+\left(1+\frac{4}{15}\pi\sigma^3 n\chi\right)\frac{m}{\theta}\sum_{\alpha,\beta=1}^{3}\left(c_\alpha c_\beta-\frac{1}{3}c^2\delta_{\alpha\beta}\right)E_{\alpha\beta}\right\}. \qquad (4.8)$$

This takes a form familiar from the kinetic theory of gases.

We can therefore transfer all the standard relationships in the kinetic theory of gases to a pseudogas.

§5. Diffusion coefficient. The solution to (1.7) found in §4 allows us to find a closed expression for the diffusion tensor in velocity space.

By definition

$$\Delta\varepsilon=-v_0\Delta n, \qquad \langle(\Delta\varepsilon)^2\rangle=v_0^2\,\langle(\Delta n)^2\rangle. \qquad (5.1)$$

The statistical properties of Δn (fluctuation in n) are completely determined by those of the set of particles forming the pseudogas, which are known, because the solution to (1.6) is known.

The mean relative fluctuation in n in any state is [13]

$$\left\langle\left(\frac{\Delta n}{n}\right)^2\right\rangle=-\frac{\theta}{v^2\partial p/\partial v}\,, \qquad (5.2)$$

in which p is pressure. It follows from §4 that

$$p=n\theta\left(1+\frac{2}{3}\pi\sigma^3 n\chi\right)-1.002\mu_0\chi\left(\frac{2}{3}\pi\sigma^3 n\right)^2\operatorname{div}\mathbf{w},$$

$$\mu_0=\frac{5}{16\sigma^2}\left(\frac{m\theta}{\pi}\right)^{1/2}. \qquad (5.3)$$

Substitution of (5.3) into (5.2) gives

$$\langle(\Delta\varepsilon)^2\rangle=v_0^2n^2\left\{1+{}^4/_3\pi\sigma^3 n\chi+{}^2/_3\pi\sigma^3 n^2\chi'-\right.$$

$$\left.-1.002\,\mu_0\theta^{-1}\operatorname{div}w\,[{}^8/_9\,\pi^2\sigma^6 n\chi+{}^4/_9\pi^2\sigma^6 n^2\chi']\right\}^{-1}. \qquad (5.4)$$

Consider now the form of $\chi(n)$, whose physical significance is the factor for the change in number of binary collisions relative to a system composed of point particles. For solid spherical particles

$$\chi\,(n)=\frac{1-{}^{11}/_{12}\pi\sigma^3 n}{1-{}^4/_3\pi\sigma^3 n}=\frac{1-{}^{11}/_{16}v_* n}{1-v_* n}=\frac{1-{}^{11}/_{16}z}{1-z}\,, \qquad (5.5)$$

in which v_* is the volume per particle in the closest spatial packing.

Substitution of (5.5) into (5.4) gives

$$\langle(\Delta\varepsilon)^2\rangle=(v_0/v_*)^2z^2(1-z)^2\left(1-z-{}^{17}/_{32}z^2+\right.$$

$$\left.+{}^{11}/_{16}z^3-{}^1/_2\mu_0 v_*\theta^{-1}\operatorname{div}w\,(z-{}^{49}/_{32}z^2+{}^{11}/_{16}z^3)\right)^{-1} \qquad (5.6)$$

Then we have

$$B_{\alpha\beta}=B_0\,(\varepsilon)\,\delta_{\alpha\beta}+B\,(q_\alpha-w_\alpha)\,(q_\beta-B_\beta) \qquad (5.7)$$

$$B=D\left(\frac{v_0}{v_*}\right)^2\Phi\left\{\frac{1}{\varepsilon}-\frac{\partial\ln\Phi}{\partial\varepsilon}\right\}^2\times$$

$$\times\ z^2(1-z)^2\left(1-z-{}^{17}/_{32}z^2+\right.$$

$$\left.+{}^{11}/_{16}z^3-{}^1/_2\mu_0 v_*\theta^{-1}\operatorname{div}w\,(z-{}^{49}/_{32}z^2+{}^{11}/_{16}z^3)\right)^{-1}. \qquad (5.8)$$

In a nonequilibrium state with $\operatorname{div}\mathbf{w}\neq 0$, $\langle(\Delta n)^2\rangle$ can increase without limit if

$$2\theta=\mu_0 v_*\frac{z-{}^{49}/_{32}z^2+{}^{11}/_{16}z^3}{1-z-{}^{17}/_{32}z^2+{}^{11}/_{16}z^3}\operatorname{div}\mathbf{w}. \qquad (5.9)$$

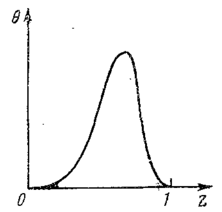
This is possible only if $(dn/dt)<0$, e.g., on marked increase in the speed of the fluid.

In the general case, $B_0(\varepsilon)$ is determined by quantities of a higher order of smallness than B, and $B_0 \ll \ll B$, so we can neglect $B_0(\varepsilon)$ when $q_\alpha - w_\alpha$ differs from zero.

We can determine θ for a steady state. From (3.8) and (3.11) we have

$$\theta = \frac{mD}{3}\left(\frac{v_0}{v_*}\right)^2\left\{\frac{1}{\varepsilon} - \frac{\partial \ln \Phi}{\partial \varepsilon}\right\}^2 \frac{z^2(1-z)^2 |\mathbf{q}-\mathbf{w}|^2}{1 - z - {}^{17}/_{32} z^2 + {}^{11}/_{16} z^3}. \quad (5.10)$$

The constant D has to be determined by experiment. The figure shows the dependence of θ on z.



§6. **Complete dynamic equations.** The results of §§4 and 5 give the following equations for the pseudogas if we substitute into (4.2) expressions for Q_α and $P_{\alpha\beta}$ in terms of the kinematic characteristics:

$$Q_\alpha = \lambda \frac{\partial \theta}{\partial x_\alpha}, \quad P_{\alpha\beta} = -p\delta_{\alpha\beta} + 2\mu e_{\alpha\beta} - 2\mu_0 S_{\alpha\beta}, \quad (6.1)$$

$$e_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial w_\alpha}{\partial x_\beta} + \frac{\partial w_\beta}{\partial x_\alpha}\right) - \frac{1}{3}\delta_{\alpha\beta} \operatorname{div} \mathbf{w}, \quad (6.2)$$

$$S_{\alpha\beta} = -\frac{m}{\theta\chi} B \times$$

$$\times \left[(q_\alpha - w_\alpha)(q_\beta - w_\beta) - \frac{1}{3}\delta_{\alpha\beta} |\mathbf{q}-\mathbf{w}|^2\right]. \quad (6.3)$$

Here λ and μ are expressed via n, χ, and θ as usual [11] for dense gases.

A major distinction of the transport equations for a pseudogas is that there is an additional term in the expressions for the components of the stress tensor, which arises from the long-range interactions between the particles and which leads to anisotropy in the properties of the system. For instance, the stress tensor is not spherical for the steady state:

$$P_{11} = -n\theta\left(1 + \frac{2}{3}\pi\sigma^3 n\chi\right) - \frac{4m\mu_0}{3\chi\theta} B |\mathbf{q}-\mathbf{w}|^2,$$

$$P_{11} - P_{22} = P_{11} - P_{33} = -\frac{2m\mu_0}{\theta\chi} B |\mathbf{q}-\mathbf{w}|^2. \quad (6.4)$$

We must consider the dynamic conditions for the motion of the liquid in order to obtain a closed system of equations describing the behavior of the two-component system.

As we are interested only in the mean characteristics of the liquid, we can use concepts concerning two-component continua, as has been done [14] for a low concentration of heavy particles in a turbulent flow. This approach has been used also for higher concentrations [2, 3], and the question of such models has been discussed in detail [15]. Similar arguments may be applied here.

Let ρ_0 be the density of the liquid, whose pressure is P and whose stress tensor is τ_{ij}. Then the equations of motion take the form

$$\frac{\partial \rho_0 \varepsilon}{\partial t} + \sum_{i=1}^{3} \frac{\partial \rho_0 \varepsilon q_i}{\partial x_i} = 0,$$

$$\rho_0 \varepsilon \frac{\partial q_i}{\partial t} + \sum_{\alpha=1}^{3} \rho_0 \varepsilon q_\alpha \frac{\partial q_i}{\partial x_\alpha} =$$

$$= -\frac{\partial P}{\partial x_i} + \sum_{j=1}^{3} \frac{\partial \tau_{ij}}{\partial x_j} + \rho_0 \varepsilon g_i - \rho\Phi(q_i - w_i). \quad (6.5)$$

In fact, τ_{ij} may be neglected for all cases of interest. The model is usable in the analysis of motion on a scale much larger than the mean distance between particles, so the viscosity of the liquid will make only a small contribution to the total tangential stress on an arbitrary area in the mixture.

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